Not Theory

Eric Nie and Alok Puranik

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Outline

What is a Knot?

Definition

Knot: an embedding of a circle into 3-space.

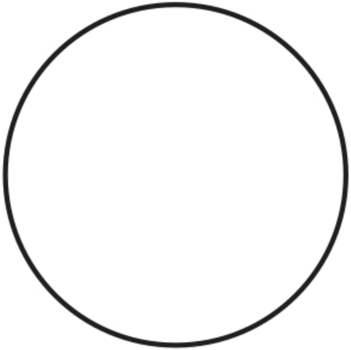
- Knots must be closed loops
- Knots must not self-intersect
- "Tame" knots must have a piecewise linear embedding

Knot Diagrams

We can model knots as pictures with knot diagrams projecting the knot into the plane with crossings

- Unknot
- Trefoil Knot
- Figure-eight Knot
- Solomon's Seal Knot

Unknot



Trefoil Knot

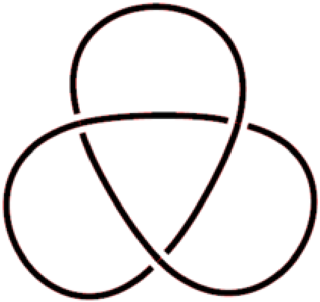
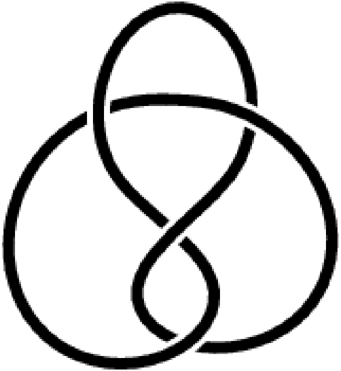
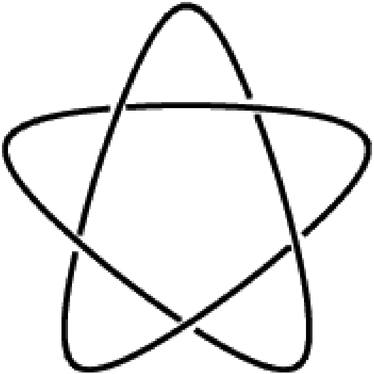


Figure-Eight Knot



Solomon's Seal Knot



The Problem

- How can we distinguish knots from the unknot?

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- How can we distinguish knots from the unknot?
- We find invariants.

Knot Sum

Definition

Connected Sum: The new knot obtained by cutting open two knots and attaching the free ends together.

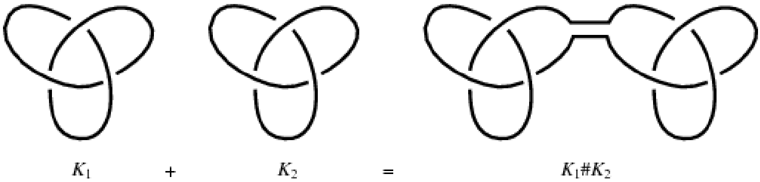


Figure: The connected sum of two knots, denoted $K_1 \# K_2$

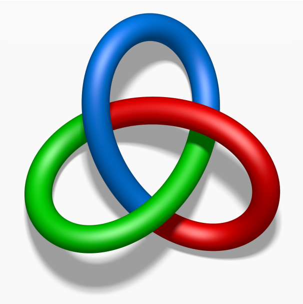
Tricolorability

Definition

Strand: an unbroken curve on a knot diagram that starts and ends at undercrossings

Definition

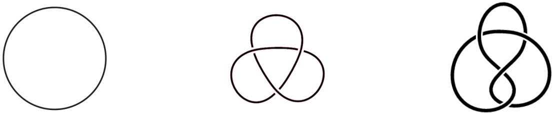
Tricolorability: a knot is tricolorable if its strands can be colored so that each crossing contains only one color or all three colors, where at least two colors are used in the diagram



Crossing Number

- Smallest number of crossings in any projection of a knot

Figure: Crossing Numbers of Knots



(a) $C(K) = 0$ (b) $C(K) = 3$ (c) $C(K) = 4$

Definition

Reduced knot diagram: one where there is no "isthmus"

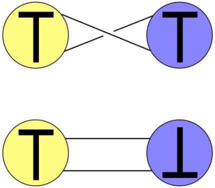


Figure: Removing an isthmus from a knot diagram

Reduced alternating knot projections demonstrate their minimum crossing numbers

Bridge Number

Definition

Minimal number of bridges required in any projection of a knot

Figure: Trefoil, drawn differently

Bridge Number

Definition

Alternate definition: the minimum number of local maxima of the knot, considered as a curve in 3-space

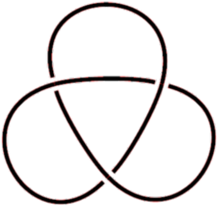


Figure: Trefoil Knot

Additivity: $Br(K_1 \# K_2) = Br(K_1) + Br(K_2) - 1$

Seifert Surfaces

Definition

An orientable surface with one boundary component, whose boundary takes the shape of a certain knot.

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Definition

An orientable surface with one boundary component, whose boundary takes the shape of a certain knot.

Construction:

1. Choose orientation for knot diagram
2. At each crossing, connect incoming strands with adjacent leaving strands
3. Move the remaining circles to different heights
4. Connect disks by twisted bands

Surface Genus

Definition

The genus of an orientable surface is the number of tori in a connected sum that is homeomorphic to it (the number of "holes" in the surface)

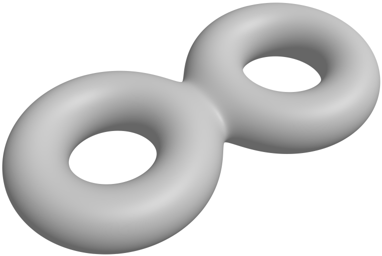


Figure: Genus 2 surface

Knot Genus

Definition

The genus of a knot is the minimal genus of any Seifert Surface for that knot.

Knot Genus is additive with respect to connected sum, so that $G(K_1 \# K_2) = G(K_1) + G(K_2)$

Arf Invariant

Definition

Pass-move: moving two oppositely oriented knot strands through two other oppositely oriented strands.

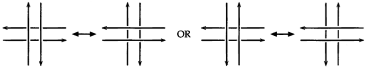


Figure: A pass-move

Definition

Knots are **pass-equivalent** if one can be converted to the other using pass-moves

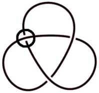
Definition

Arf invariant: 1 if a knot is pass equivalent to the trefoil knot, 0 if equivalent to the unknot

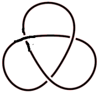
Jones Polynomial

- Skein Relation:
- $t^{-1}V(L_+) - tV(L_-) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V(L_0) = 0$

Figure: Skein Relation

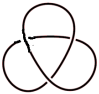


(a) Trefoil Knot



(b) L_+

(c) L_0



(d) L_-

Alexander Polynomial

Rules

- $\Delta(O) = 1$
- $\Delta(L_+) - \Delta(L_-) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(L_0) = 0$

HOMFLY Polynomial

Rules

- $P(O) = 1$
- $lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0$

Acknowledgements

- MIT PRIMES
- Zhenkun Li
- Our parents